A COMBINATORIAL PROBLEM AND ITS APPLICATION TO PROBABILITY THEORY—I

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Summary

We solve a combinatorial problem which generalizes the 'problème du scrutin' of D. André. In a particular case, this result may be interpreted as a quasi-order defined on the r-partitions of an integer. We indicate the relation of these quasi-orderings to certain coin tossing problems in probability theory considered by the author in a previous paper.

Notations

We follow closely the notations and methods used by the author in a previous paper 'On the lattices formed by the partitions of an integer and their application to probability theory' [1].

1. Statement of Problem

Suppose that we are given k sets of balls, the ith set consisting of a_i balls $(i = 1, \ldots, k)$. Corresponding to each set of balls we are given an ordered set of r boxes, the r boxes corresponding to the ith set of balls being numbered $i_1, i_2, \ldots, i_r$ $(i = 1, \ldots, k)$. We shall suppose that $a_i \geqslant r$ for all i. Let further $(k-1)$ non-negative integers $L_1, L_2, \ldots, L_{k-1}$ be given, satisfying the conditions

$$a_i + L_i \geqslant a_{i+1} \quad i = 1, \ldots, k-1. \tag{1}$$

We distribute each set of balls in the corresponding set of r boxes, so that no box is empty and so that the following further condition is satisfied:

If $t_1^{(i)}, t_2^{(i)}, \ldots, t_r^{(i)}$ be the number of balls of the ith set in the boxes $i_1, i_2, \ldots, i_r$ respectively $(i = 1, \ldots, k)$, then

$$\left.\begin{array}{c} t_1^{(i)} + L_i \geqslant t_1^{(i+1)} \\ t_1^{(i)} + t_2^{(i)} + L_i \geqslant t_1^{(i+1)} + t_2^{(i+1)} \\ \vdots \qquad\qquad \vdots \\ t_1^{(i)} + \ldots + t^{(i)}_{r-1} + L_i \geqslant t_1^{(i+1)} + t_2^{(i+1)} + \ldots + t_{r-1}^{(i+1)} \\ t_1^{(i)} + \ldots + t_r^{(i)} + L_i \geqslant t_1^{(i+1)} + \ldots + t_r^{(i+1)} \end{array}\right\} \tag{2}$$

for $i = 1, \ldots, k-1$.

We note that the last inequality of (2) is satisfied by virtue of (1). The $t^{(i)}$'s are all integral and greater than or equal to unity and $t_1^{(i)} + \ldots + t_r^{(i)} = a_i$ for all $i = 1, \ldots, k$.

We shall state and prove Theorem 1 which gives us the total number of ways of distributing the balls in the boxes under the above conditions.

Theorem 1.—*The total number of ways of distributing the k sets of balls in the corresponding sets of boxes, satisfying the aforementioned conditions is,*

$$(a_1, a_2, \ldots, a_k)_{r,0}\ [L_1, L_2, \ldots, L_{k-1}]$$

$$= \begin{vmatrix} (a_1-1)_{(r-1)} & (a_2-1-L_1)_{(r)} \ldots & (a_k-1-\overline{L_1+\ldots+L_{k-1}})_{(r+k-2)} \\ (a_1-1+L_1)_{(r-2)} & (a_2-1)_{(r-1)} \ldots & (a_k-1-\overline{L_2+\ldots+L_{k-1}})_{(r+k-3)} \\ (a_1-1+L_1+L_2)_{(r-3)} & (a_2-1+L_2)_{(r-2)} \ldots & (a_k-1-\overline{L_3+\ldots+L_{k-1}})_{(r+k-4)} \\ \vdots & \vdots & \vdots \\ (a_1-1+L_1+\ldots+L_{k-1})_{(r-k)} & (a_2-1+L_2+\ldots+L_{k-1})_{(r-k+1)} \ldots & (a_k-1)_{(r-1)} \end{vmatrix} \quad (3)$$

where

$$(a_i - 1)_{(t)} = {}^{(a_i-1)}C_t.$$

2. Some Properties of Certain Auxiliary Determinants

We shall consider some properties of the determinants $(a_1, a_2, \ldots, a_k)_{r,0}$ and $(a_1, a_2, \ldots, a_k)_{r,t}$ which we define below, to prove Theorem 1.

We denote by $(a_1, a_2, \ldots, a_k)_{r,0}$ the determinant of the kth order obtained from (3) by setting $L_1 = L_2 = \ldots = L_{k-1} = 0$; *i.e.*,

$$(a_1, a_2, \ldots, a_k)_{r,0} = \begin{vmatrix} (a_1-1)_{(r-1)} & (a_2-1)_{(r)} & \ldots & (a_k-1)_{(r+k-2)} \\ (a_1-1)_{(r-2)} & (a_2-1)_{(r-1)} & \ldots & (a_k-1)_{(r+k-3)} \\ \vdots & \vdots & & \vdots \\ (a_1-1)_{(r-k)} & (a_2-1)_{(r-k+1)} & \ldots & (a_k-1)_{(r-1)} \end{vmatrix}. \quad (4)$$

We denote by $(a_1, a_2, \ldots, a_k)_{r,t}$ the determinant, given below, obtained from $(a_1, a_2, \ldots, a_k)_{r,0}$ by adding t to each of the subscripts in the first row of $(a_1, a_2, \ldots, a_k)_{r,0}$; *i.e.*,

$$(a_1, a_2, \ldots, a_k)_{r,t} = \begin{vmatrix} (a_1-1)_{(r+t-1)} & (a_2-1)_{(r+t)} & \ldots & (a_k-1)_{(r+k+t-2)} \\ (a_1-1)_{(r-2)} & (a_2-1)_{(r-1)} & \ldots & (a_k-1)_{(r+k-3)} \\ \vdots & \vdots & & \vdots \\ (a_1-1)_{(r-k)} & (a_2-1)_{(r-k+1)} & \ldots & (a_k-1)_{(r-1)} \end{vmatrix}. \quad (5)$$

We note the following properties of the determinants given in (4) and (5) for $k \geqslant 2$: if

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k \geqslant r+1, \quad (6)$$

then

$$\sum_{a_1' \geqslant a_2'}^{a_1-1} \sum_{a_2' \geqslant a_3'}^{a_2-1} \cdots \sum_{a'_{k-1} \geqslant a_k'}^{a_{k-1}-1} \sum_{a_k'=1}^{a_k-1} (a_1', a_2', \ldots, a_k')_{r,0} = (a_1, a_2, \ldots, a_k)_{r+1,0}. \quad (7)$$

We note first that if (6) is not satisfied, (7) is still trivially valid, since the summands reduce to zero. From (1), it is clear that in the case $L_1 = \ldots = L_{k-1} = 0$, $(a_1, \ldots, a_k)_{r,0}$ is non-zero, if and only if, $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k \geqslant r$.

We establish (7) for the case $k = 3$, the general result being similar. Consider,

$$\sum_{a_1' \geqslant a_2'}^{a_1-1} \sum_{a_2' \geqslant a_3'}^{a_2-1} \sum_{a_3' \geqslant 1}^{a_3-1} \begin{vmatrix} (a_1'-1)_{(r-1)} & (a_2'-1)_{(r)} & (a_3'-1)_{(r+1)} \\ (a_1'-1)_{(r-2)} & (a_2'-1)_{(r-1)} & (a_3'-1)_{(r)} \\ (a_1'-1)_{(r-3)} & (a_2'-1)_{(r-2)} & (a_3'-1)_{(r-1)} \end{vmatrix}.$$

Summing over a_1', we obtain $(a_1 - a_2')$ determinants of the same form as above, with the same 2nd and 3rd columns. Hence the result of the summation over a_1' is

$$\sum_{a_2' \geqslant a_3'}^{a_2-1} \sum_{a_3' \geqslant 1}^{a_3-1} \begin{vmatrix} (a_1-1)_{(r)} - (a_2'-1)_{(r)} & (a_2'-1)_{(r)} & (a_3'-1)_{(r+1)} \\ (a_1-1)_{(r-1)} - (a_2'-1)_{(r-1)} & (a_2'-1)_{(r-1)} & (a_3'-1)_{(r)} \\ (a_1-1)_{(r-2)} - (a_2'-1)_{(r-2)} & (a_2'-1)_{(r-2)} & (a_3'-1)_{(r-1)} \end{vmatrix}.$$

or,

$$\sum_{a_2' \geqslant a_3'}^{a_2-1} \sum_{a_3' \geqslant 1}^{a_3-1} \begin{vmatrix} (a_1-1)_{(r)} & (a_2'-1)_{(r)} & (a_3'-1)_{(r+1)} \\ (a_1-1)_{(r-1)} & (a_2'-1)_{(r-1)} & (a_3'-1)_{(r)} \\ (a_1-1)_{(r-2)} & (a_2'-1)_{(r-2)} & (a_3'-1)_{(r-1)} \end{vmatrix}.$$

Continuing next the summation over a_2', and then over a_3' (using the same method), we have easily,

$$\sum_{a_1' \geqslant a_2'}^{a_1-1} \sum_{a_2' \geqslant a_3'}^{a_2-1} \sum_{a_3' \geqslant 1}^{a_3-1} (a_1', a_2', a_3')_{r,0} = (a_1, a_2, a_3)_{r+1,0}.$$

We have, in fact, the much stronger result, which can be established similarly:

$$\sum_{a_1' \geqslant a_2'}^{a_1-1} \sum_{a_2' \geqslant a_3'}^{a_2-1} \cdots \sum_{a_{k-1}' \geqslant a_k'}^{a_{k-1}-1} \sum_{a_k'=1}^{a_k-1} (a_1', \ldots, a_k')_{r,t} = (a_1, \ldots, a_k)_{r+1,t}. \quad (8)$$

We now prove the identity:

$$\sum_{a_1' \geqslant a_2'}^{a_1-1} \sum_{a_2' \geqslant a_3'}^{a_2-1} \cdots \sum_{a_k' \geqslant 1}^{a_k-1} (a_1'-1)_{(s)} = (a_1, a_2, \ldots, a_k)_{2,s}, \quad (9)$$

where

$$(a_1'-1)_{(s)} = {}^{(a_1'-1)}C_s.$$

Let us consider (9) when $k = 2$.

Obviously,

$$\sum_{a_1' \geqslant a_2'}^{a_1-1} \sum_{a_2'=1}^{a_2-1} (a_1'-1)_{(s)} = \sum_{a_2'=1}^{a_2-1} \{(a_1-1)_{(s+1)} - (a_2'-1)_{(s+1)}\}$$

$$= \begin{vmatrix} (a_1-1)_{(s+1)} & (a_2-1)_{(s+2)} \\ (a_1-1)_{(0)} & (a_2-1) \end{vmatrix}$$

$$= (a_1, a_2)_{2,s}.$$

Thus (9) is proved when $k = 2$.

Now

$$\sum_{a_1'\geqslant a_2'}^{a_1-1} \sum_{a_2'\geqslant a_3'}^{a_2-1} \sum_{a_3'\geqslant 1}^{a_3-1} (a_1'-1)_{(s)}$$

$$= \sum_{a_2'\geqslant a_3'}^{a_2-1} \sum_{a_3'\geqslant 1}^{a_3-1} \{(a_1-1)_{(s+1)} - (a_2'-1)_{(s+1)}\}$$

$$= (a_1-1)_{(s+1)} \sum_{a_2'\geqslant a_3'}^{a_2-1} \sum_{a_3'\geqslant 1}^{a_3-1} 1 - \sum_{a_2'\geqslant a_3'}^{a_2-1} \sum_{a_3'\geqslant 1}^{a_3-1} (a_2'-1)_{(s+1)}$$

$$= (a_1-1)_{(s+1)} (a_2, a_3)_{2,0} - (a_2, a_3)_{2,s+1},$$

since (9) was proved for $k = 2$.

Noting that:

$$(a_1-1)_{(s+1)} (a_2, a_3)_{2,0} - (a_2, a_3)_{2,s+1}$$

$$= \begin{vmatrix} (a_1-1)_{(s+1)} & (a_2-1)_{(s+2)} & (a_3-1)_{(s+3)} \\ 0 & (a_2-1)_{(1)} & (a_3-1)_{(2)} \\ 0 & (a_2-1)_{(0)} & (a_3-1)_{(1)} \end{vmatrix} + \begin{vmatrix} 0 & (a_2-1)_{(s+2)} & (a_3-1)_{(s+3)} \\ 1 & (a_2-1)_{(1)} & (a_3-1)_{(2)} \\ 0 & (a_2-1)_{(0)} & (a_3-1)_{(1)} \end{vmatrix}$$

[on suitably bordering the determinants]

$$= (a_1, a_2, a_3)_{2,s},$$

we have proved (9) when $k = 3$.

By induction, (9) can be shown valid for all k.

Other properties of the determinants $(a_1, a_2, \ldots, a_k)_{r,t}$ can be derived similarly.

3. Proof of Theorem 1

We shall prove Theorem 1 in the case $L_1 = L_2 = \ldots = L_{k-1} = 0$, since the general case is analogous. In this case, Theorem 1 was stated in [1] and proved for $k = 2$ using a geometrical interpretation. We shall use the same method and interpretation in what follows.

In order to prove Theorem 1, when $L_1 = L_2 = \ldots = L_{k-1} = 0$, we first remark that the case $k = 1$ is immediate. The case $k = 2$ was proved in [1]. Let us consider the case $k = 3$.

Case $k = 3$.—When $r = 1$, the proof is evident. Let us consider the case $r = 2$. Given that a particle starting from the origin has reached the point $P\,(a_1, a_2, a_3)$ $(a_1 \geqslant a_2 \geqslant a_3 \geqslant 2)$ in two steps under condition (2), the number of ways in which this could have happened is evidently

$$\sum_{a_1 \geqslant a_2}^{a_1-1} \sum_{a_2 \geqslant a_3}^{a_2-1} \sum_{a_3 \geqslant 1}^{a_3-1} 1 = (a_1, a_2, a_3)_{2,0},$$

by identity (9) where we take $s = 0$. Thus the case $k = 3$, $r = 2$ is proved.

But we know that if the particle reached $P(a_1, a_2, a_3)$ in 3 steps, the number of ways in which this could happen is

$$\sum_{a_1 \geqslant a_2}^{a_1-1} \sum_{a_2 \geqslant a_3}^{a_2-1} \sum_{a_3 \geqslant 1}^{a_3-1} (a_1, a_2, a_3)_{2,0}.$$

Using (7) when $r = 2$, $k = 3$, the summation in the last line equals $(a_1, a_2, a_3)_{3,0}$ so that the case $k = 3$, $r = 3$ is proved.

Proceeding recursively, using the same argument and equation (7) for a suitable value of r, the case $k = 3$ can be proved for a general r.

The proof of Theorem 1 for a general k is evident in the special case $L_1 = \ldots = L_{k-1} = 0$, making a double induction on k and r. Let us suppose that the special case of Theorem 1 is proved for all values up to and including k for all r. We consider now the case for $k + 1$ sets of balls containing $a_1, a_2, \ldots, a_{k+1}$ balls. For $k + 1$ sets, when $r = 1$, the proof is evident. Using equation (9) when $s = 0$, the theorem is proved for $k + 1$ sets of balls and $r = 2$. The geometrical interpretation and equation (7) permit us to conclude the validity of the theorem for $k + 1$ sets of balls and a general r. Hence the proof of Theorem 1 is complete, when $L_1 = \ldots = L_{k-1} = 0$.

We now remark that equations similar to (7), (8) and (9) can be established for the determinant denoted by $(a_1, a_2, \ldots, a_k)_{r,0}$ $[L_1, L_2, \ldots, L_{k-1}]$, where $L_1, \ldots, L_{k-1}$ are non-negative integers. The modifications to be made in the summations and geometrical interpretation are trivial. Hence Theorem 1 is proved in the general case as well.

4. Interpretations of Theorem 1

(*a*) Let us consider the case $k = 2$ of Theorem 1, when $a_1 = a_2 = n$ (say) and $L_1 = L$ say. Equation (1) is trivially satisfied and equations (2) can be written:

$$\left.\begin{array}{c} t_1^{(1)} + L \geqslant t_1^{(2)} \\ t_1^{(1)} + t_2^{(1)} + L \geqslant t_1^{(2)} + t_2^{(2)} \\ \vdots \\ t_1^{(1)} + \ldots + t_{r-1}^{(1)} + L \geqslant t_1^{(2)} + \ldots + t_{r-1}^{(2)} \\ n + L \geqslant n \end{array}\right\} \quad (10)$$

where

$$t_i^{(j)} \geqslant 1 \ (i = 1, \ldots, r; \ j = 1, 2),$$
$$t_1^{(1)} + \ldots + t_r^{(1)} = t_1^{(2)} + \ldots + t_r^{(2)} = n,$$

and

$$(t_1^{(1)}, \ldots, t_r^{(1)}), \ (t_1^{(2)}, \ldots, t_r^{(2)})$$

are r-partitions of n in the notation of [1]. When $L > 0$, the relations (10) represent a *quasi-order* defined on the r-partitions of n, since they are obviously reflexive and transitive. In analogy with [1], we call this quasi-order the relation of 'L-domination', or we say that the r-partition of n $(t_1^{(1)}, \ldots, t_r^{(1)})$ dominates (L) the r-partition of n $(t_1^{(2)}, \ldots, t_r^{(2)})$. Most of the ideas expressed for the case $L = 0$ in [1] generalise for the case $L > 0$ as well.

When $L = 0$, the relations (10) represent a partial order defined on the r-partitions of n [1]. If in the relations (10) the inequality sign were strict [except for the last line of (10)] and $L = 0$, we would obtain the relation of strict domination as opposed to that of domination. These relations would correspond to the recurrent events of Feller for coin-tossing in its simplest case (*cf.* [2]). A result similar to Theorem 1 can be obtained for strict dominations as well.

The case $L < 0$ is worthy of note. The author has obtained results similar to the cases $L \geqslant 0$, and a generalization of Theorem 1 where we now allow $L_1, \ldots, L_{k-1}$ to take positive or negative integral values. However the relations (10) when $L > 0$ are not reflexive though they continue to be transitive. {The relations (10′) obtained from (10) where we replace the $\geqslant$ sign by a $\leqslant$ sign and $L < 0$, would lead us again to the same quasi-order as for the case $L > 0$. This is a general situation which would arise in all partial or quasi-orders [3]}.

(*b*) Let $k = 2$, $a_1 = m$, $a_2 = n$, $L_1 = L$. The relations (2) would then enable us to see whether an r-partition of m dominates (L) an r-partition of n. Let us suppose that we number the $\binom{m-1}{r-1}$ r-partitions of m using the symbols $p_1, p_2, \ldots, p_{\binom{m-1}{r-1}}$ and similarly number the r-partitions of n $p_1', p_2', \ldots, p'_{\binom{n-1}{r-1}}$. Let n_i denote the number of r-partitions in the set $p_1', p_2', \ldots, p'_{\binom{n-1}{r-1}}$ dominated (L) by p_i, $i = 1, 2, \ldots, \binom{m-1}{r-1}$. The sum $(m, n)_r^L = n_1 + \ldots + n_{\binom{m-1}{r-1}}$ is evidently independent of the numbering chosen for the r-partitions of m and n, and depends only on m, n, L, r. We have as a corollary of Theorem 1 that,

$$(m, n)_r^L = (m-1)_{(r-1)} (n-1)_{(r-1)} - (m+L-1)_{(r-2)} (n-L-1)_{(r)}.$$

Let us further set $n = m + L$. With a change of notation, we obtain the useful result:

$$(n, n+k)^k{}_r = (n-1)_{(r-1)} (n+k-1)_{(r-1)} - (n+k-1)_{(r-2)} (n-1)_{(r)}. \quad (11)$$

(*c*) The modifications required to prove results similar to Theorem 1 when we define an r-partition of n as a set of t_i, where $t_i \geqslant 0$ for $i = 1, \ldots, r$ so that

$$t_1 + \ldots + t_r = n$$

are obvious. The author is investigating the case where the relation (10) is replaced by the following:—

$$t_1^{(1)} + l_1 \geqslant t_1^{(2)}$$

$$t_1^{(1)} + t_2^{(1)} + l_2 \geqslant t_1^{(2)} + t_2^{(2)}$$

$$\vdots \qquad \vdots$$

$$t_1^{(1)} + t_2^{(1)} + \ldots + t_{r-1}^{(1)} + l_{r-1} \geqslant t_1^{(2)} + t_2^{(2)} + \ldots + t_{r-1}^{(2)}.$$

5. Application to the Theory of Probability

Let us suppose that we are given two coins 1, 2 with probabilities p_1, p_2 of obtaining heads and, consequently the probabilities q_1, q_2 of obtaining tails where $q_i = 1 - p_i$, $i = 1, 2$. We shall assume in what follows that $p_1 + p_2 > 1$.

Let us consider the game G_n $[n \geqslant 2]$ played with the following rules:

(1) The first trial is made with coin 1.

(2) For $n > 1$, the nth trial is made with coin 1 or coin 2, according as the result of the $(n-1)^{st}$ trial was a tail or head.

(3) We stop the series of trials at that trial where for the first time the accumulated number of heads obtained (with both coins) is greater than the accumulated number of tails obtained by exactly n.

When $n = 2$, this problem was considered and solved in [1]. The methods and results obtained in [1] can be generalized easily to obtain the solution of the game G_n for $n \geqslant 3$. Using the notations of [1] where base sequences and S_r are defined, we state the following Theorem 2:

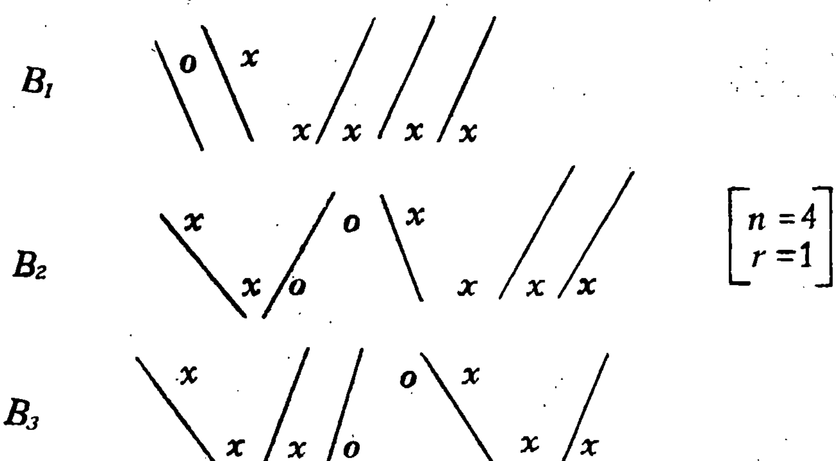
Theorem 2.—*The number of base sequences of* G_n $(n \geqslant 3)$ *in* S_r $(r \geqslant 1)$ *containing* $(n + 2r + 2t - 2)$ *terms is*

$$(r+1,\ n+r-2)_t^{n-3} = r_{(t-1)}\,(n+r-3)_{(t-1)} - (n+r-3)_{(t-2)}\,r_{(t)}$$

where

$$t = 1, \ldots, r+1.$$

As an illustration of Theorem 2, we give the base sequences of G_4 in S_1 below. They are



B_1 contains 6 trials and B_2, B_3 consist of 8 trials each. B_1 represents a domination (1) of the 1-partition of 3 by the 1-partition of 2. B_2, B_3 correspond to the fact that the partition (1, 1) of 2 dominates (1) the partitions (1, 2), (2, 1) of 3. The sloping lines \ and /, which indicate the positions where the subsidiary sequences x_0, ${}_0x$ could be introduced, demonstrate the possible 1-dominations of the partitions of 3 by those of 2.

Following [1] we thus obtain the identity

$$\sum_{r=0}^{\infty} \frac{q_1^r p_1 p_2^{n+r-1}}{(1-p_1q_2)^{n+2r-1}} \frac{1}{n+r-2} \sum_{t=0}^{r} (n+r-2)_{(t)} \times [(n-2)\,r_{(t)} + r_{(t+1)}]\,(p_1 q_2)^t = 1,$$

for integral $n \geqslant 3$, where $p_i + q_i = 1$, $i = 1, 2$ and $p_1 + p_2 > 1$.

We finally state that Theorem 1 can be applied to other kinds of coin tossing experiments with a similar stopping rule. The game G_1, which requires a special consideration, and other similar coin tossing experiments will be discussed in detail in a further paper.

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REFERENCES

1. Narayana, T. V. .. "Sur les treillis formés par les partitions d'un entier et leurs applications à la théorie des probabilités", *Comptes Rendus*, *t.* 240, pp. 1188–89.

2. Feller, W. .. *An Introduction to Probability Theory and its Applications.*

3. Birkhoff, G. .. *Lattice Theory.*